Currents and charges for mixed fields

Massimo Blasone[♯♭], Petr Jizba[♮] and Giuseppe Vitiello[♭]

[♯] *Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, U.K.*

[♮] *Institute of Physics, University of Tsukuba, Ibaraki 305-8571, Japan*

[♭] *Dipartimento di Fisica and INFN, Università di Salerno, I-84100 Salerno, Italy*

(**Version**, February 1, 2008)

We present an analysis of currents and charges for a system of two mixed fields, both for spinless bosons and for Dirac fermions. This allows us to obtain in a straightforward way the exact field theoretical oscillation formulas exhibiting corrections with respect to the usual ones derived in quantum mechanics.

I. INTRODUCTION

Despite being studied since long time, particle mixing and oscillations [1–3] are still among the most fascinating subjects of Particle Physics: there is a renewed interest in the physics of mixed mesons (like Kaons or B-mesons) due to the ongoing or planned experiments [4] and neutrino oscillations [5] seem finally to have been observed [6] after a long search. On the other hand, on the theoretical side, many questions remain unanswered like the ones on the origin itself of the mixing among generations or about the values of the lepton mixing angles [7]. Also, various issues related to the phenomenology of flavor oscillations are object of active discussions (see Refs. [3,8] for recent reviews).

Recently, there has been a remarkable progress in the understanding of the Quantum Field Theory (QFT) of mixed particles, for both fermions and bosons [9–16]. It has emerged [9,10] that the vacuum state for the flavor fields (describing the mixed particles) is not the same as the one for the mass eigenstates, rather is a generalized coherent state [17]. This fact turns out to have observable consequences: the exact oscillation formulas obtained in QFT for neutrinos [11,13] and for bosons [16] exhibit corrections with respect to the usual textbook expressions derived in Quantum Mechanics (QM) [1,2].

The aim of this paper is to present an analysis of the currents and the charges for a system of mixed fields, which can then be used for deriving oscillation formulas. We consider both spin zero charged bosons and Dirac fermions in the simplest case of mixing among two generations.

The fact that such a problem has remained basically unsolved till now is due to the fundamental difficulties one encounters in the definition of the Hilbert space for particles with undefined mass, as the mixed particles are. Thus the results of Refs. [9,16] are crucial in the following discussion. To our knowledge, the only existing treatment of currents and charges for mixed particles is the one of Ref. [18]: the results there obtained are valid in the framework of non-relativistic Quantum Mechanics, and therefore have to be considered only approximate (i.e. they should be regarded as the non-relativistic limit of exact ones). This is so because mixing between states of different masses is not even allowed in non-relativistic QM due to superselection rules [19].

Apart from the intrinsic theoretical interest of our analysis, its necessity has been recently emphasized in Ref. [3]: an unambiguous definition of currents and charges for mixed particles would lead to a more consistent way to treat particle oscillations, especially for the case (as it happens for actual experiments) in which the time of measurement is not known and what is really measured is the distance from the source. In Ref. [18], oscillation formulas for the case of Kaon mixing are obtained after integrating the probability current density over the time of measurement and the surface of the detector (L denotes the distance from the source and α, β are flavor indices):

$$P_{\alpha\rightarrow\beta}(L) = \int_{t_1}^{t_2} dt \int_{\partial A} \mathbf{dS} \cdot \mathbf{j}_\beta(x,t)\,,$$

from which results in line with the usual formulas follow.

In QFT the situation is different: rather than to probability current, one has to look at expectation values of operators describing the flavor currents and charges. These expectation values have to be properly defined on the Hilbert space for mixed particles which cannot be identified with the space for the mass eigenstates, as shown in Refs. [9,10,16]. What we show here is that currents and charges for a system of two mixed fields can be indeed properly defined in QFT: these are the currents associated with $SU(2)$ transformations acting on the field doublet. By using the flavor Hilbert space [9,16], we then derive exact QFT oscillation formulas (both for fermions and bosons) which agree with the ones already presented in Refs. [11,13,16]. We do not discuss here oscillation formulas in space, although our results naturally provide a basis for such calculation, which will be treated elsewhere.

II. CURRENTS AND CHARGES FOR MIXED FIELDS

Let us begin our discussion with the fermion case and consider the following Lagrangian density describing two Dirac fields with a mixed mass term (in the following we will refer explicitly to neutrinos):

$$\mathcal{L}(x) \,=\, \bar{\Psi}_f(x)\,(i\,\not\partial - M)\,\Psi_f(x)\,, \qquad (1)$$

where $\Psi_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$. This Lagrangian has been already used in Ref. [11] to model neutrino oscillations in vacuo.

The mixing transformations

$$\Psi_f(x) \,=\, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Psi_m(x)\,, \qquad (2)$$

with θ being the mixing angle and $\Psi_m^T = (\nu_1, \nu_2)$, diagonalize the quadratic form of Eq.(1) to the Lagrangian for two free Dirac fields, with masses m_1 and m_2:

$$\mathcal{L}(x) \,=\, \bar{\Psi}_m(x)\,(i\,\not\partial - M_d)\,\Psi_m(x)\,, \qquad (3)$$

where $M_d = diag(m_1, m_2)$. One also has $m_e = m_1\cos^2\theta + m_2\sin^2\theta$, $m_\mu = m_1\sin^2\theta + m_2\cos^2\theta$, $m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta$.

The free fields ν_i (i=1,2) can be quantized in the usual way [20] (we use $t \equiv x_0$):

$$\nu_i(x) = \sum_r \int d^3k \left[u^r_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}(t) \,+ v^r_{-\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (4)$$

with $\alpha^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}\alpha^r_{\mathbf{k},i}(0)$, $\beta^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}\beta^r_{\mathbf{k},i}(0)$ and $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The vacuum for the mass eigenstates is denoted by $|0\rangle_{1,2}$: $\alpha^r_{\mathbf{k},i}|0\rangle_{1,2} = \beta^r_{\mathbf{k},i}|0\rangle_{1,2} = 0$. The anticommutation relations are the usual ones; the wave function orthonormality and completeness relations are those of Ref. [9].

We now study the transformations acting on the doublet of free fields with different masses. $\mathcal{L}$ is invariant under global $U(1)$ phase transformations of the type $\Psi'_m = e^{i\alpha}\,\Psi_m$: as a result, we have the conservation of the Noether charge $Q = \int d^3x\, I^0(x)$ (with $I^\mu(x) = \bar{\Psi}_m(x)\,\gamma^\mu\,\Psi_m(x)$) which is indeed the total charge of the system (i.e. the total lepton number).

Consider now the $SU(2)$ transformations acting on Ψ_m:

$$\Psi'_m(x) \,=\, e^{i\alpha_j\tau_j}\,\Psi_m(x) \qquad , \qquad j = 1,2,3. \qquad (5)$$

with α_j real constants, $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

For $m_1 \neq m_2$, the Lagrangian is not generally invariant under (5) and we obtain, by use of the equations of motion,

$$\delta\mathcal{L}(x) \,=\, i\alpha_j\,\bar{\Psi}_m(x)\,[\tau_j, M_d]\,\Psi_m(x) \,=\, -\alpha_j\,\partial_\mu J^\mu_{m,j}(x)$$

$$J^\mu_{m,j}(x) \,=\, \bar{\Psi}_m(x)\,\gamma^\mu\,\tau_j\,\Psi_m(x) \qquad , \qquad j = 1,2,3. \quad (6)$$

The charges $Q_{m,j}(t) \equiv \int d^3x\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra $[Q_{m,j}(t), Q_{m,k}(t)] \,=\, i\,\epsilon_{jkl}\,Q_{m,l}(t)$. Note that the Casimir operator is proportional to the total charge: $C_m \equiv \left[\sum_{j=1}^{3} Q^2_{m,j}(t)\right]^{\frac{1}{2}} = \frac{1}{2}Q$. From (6) we also see that $Q_{m,3}$ is conserved as M_d is diagonal. We can define the combinations:

$$Q_1 \,\equiv\, \frac{1}{2}Q \,+\, Q_{m,3} \quad , \quad Q_2 \,\equiv\, \frac{1}{2}Q \,-\, Q_{m,3} \qquad (7)$$

$$Q_i \,=\, \sum_r \int d^3k\,\left(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} \,-\, \beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},i}\right) \quad , \quad i = 1,2.$$

These are nothing but the Noether charges associated with the non-interacting fields ν_1 and ν_2: in the absence of mixing, they are the flavor charges, separately conserved for each generation.

Observe now that the transformation:

$$\Psi_f(x) = e^{-2i\theta Q_{m,2}(t)}\Psi_m(x)e^{2i\theta Q_{m,2}(t)} \qquad (8)$$

is just the mixing Eq.(2). Thus $2Q_{m,2}(t)$ is the generator of the mixing transformations: its properties have been studied in ref. [9], where it has been shown that the action of $G_\theta(t) \equiv e^{2i\theta Q_{m,2}(t)}$ on the vacuum state $|0\rangle_{1,2}$ results in a new vector (flavor vacuum) $|0(t)\rangle_{e,\mu} \equiv G^{-1}_\theta(t)|0\rangle_{1,2}$, orthogonal to $|0\rangle_{1,2}$ in the infinite volume limit.

By use of $G_\theta(t)$, the flavor fields can be expanded as:

$$\nu_\sigma(x) = \sum_r \int d^3k\,\left[u^r_{\mathbf{k},i}\alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}}\,.$$

with $(\sigma, i) = (e,1),(\mu,2)$. The flavor annihilation operators are defined as $\alpha^r_{\mathbf{k},\sigma}(t) \equiv G^{-1}_\theta(t)\alpha^r_{\mathbf{k},i}(t)G_\theta(t)$ and $\beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \equiv G^{-1}_\theta(t)\beta^{r\dagger}_{-\mathbf{k},i}(t)G_\theta(t)$.

Let us now perform the $SU(2)$ transformations on the flavor doublet Ψ_f:

$$\Psi'_f(x) \,=\, e^{i\alpha_j\tau_j}\,\Psi_f(x) \qquad , \qquad j \,=\, 1,2,3. \qquad (9)$$

We obtain

$$\delta\mathcal{L}(x) \,=\, i\alpha_j\,\bar{\Psi}_f(x)\,[\tau_j, M]\,\Psi_f(x) \,=\, -\alpha_j\,\partial_\mu J^\mu_{f,j}(x)\,,$$

$$J^\mu_{f,j}(x) \,=\, \bar{\Psi}_f(x)\,\gamma^\mu\,\tau_j\,\Psi_f(x) \qquad , \qquad j \,=\, 1,2,3. \quad (10)$$

The related charges $Q_{f,j}(t) \equiv \int d^3x\, J^0_{f,j}(x)$, still close the $su(2)$ algebra and $C_f = C_m = \frac{1}{2}Q$. Due to the off–diagonal (mixing) terms in the mass matrix M, $Q_{f,3}(t)$ is time–dependent. This implies an exchange of charge between ν_e and ν_μ, resulting in the flavor oscillations.

In accordance with Eq.(7), we define the *flavor charges* for mixed fields as

$$Q_e(t) \equiv \frac{1}{2}Q \,+\, Q_{f,3}(t) \qquad (11)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q \,-\, Q_{f,3}(t) \qquad (12)$$

with $Q_e(t) + Q_\mu(t) = Q$.

These charges have a simple expression in terms of the flavor ladder operators ($\sigma = e, \mu$):

$$Q_\sigma(t) = \sum_r \int d^3k \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right) ,$$

which happens because they are connected to the Noether charges Q_i of Eq.(7) via the mixing generator: $Q_\sigma(t) = G^{-1}_\theta(t) Q_i G_\theta(t)$. Notice also that the operator

$$\Delta Q_e \equiv Q_e(t) - Q_1 \,=\, Q_{f,3}(t) - Q_{m,3} \qquad (13)$$

describes how much the mixing violates the lepton number conservation for a given generation.

The oscillation formulas are obtained by taking expectation values of the above charges on the (flavor) neutrino state. Consider for example an initial electron neutrino state defined as $|\nu_e\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},e}(0)|0\rangle_{e,\mu}$ (for a discussion on the correct definition of flavor states see Refs. [11,13,16]). Working in the Heisenberg picture, we obtain

$$ {}_{e,\mu}\langle 0|Q_\sigma(t)|0\rangle_{e,\mu} \,=\, 0 \,, \qquad (14)$$

$$\begin{aligned} \mathcal{Q}_{\mathbf{k},\sigma}(t) \,&\equiv\, \langle \nu_e|Q_\sigma(t)|\nu_e\rangle \qquad (15)\\ &= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 \,+\, \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 . \end{aligned}$$

where $|0\rangle_{e,\mu} \equiv |0(0)\rangle_{e,\mu}$. Charge conservation is obviously ensured at any time: $\mathcal{Q}_{\mathbf{k},e}(t) + \mathcal{Q}_{\mathbf{k},\mu}(t) = 1$. We remark that the expectation value of Q_σ cannot be taken on vectors of the Fock space built on $|0\rangle_{1,2}$, as shown in Refs. [11,13,16].

The oscillation (in time) formulas for the flavor charges, on an electron neutrino state, then follow:

$$\begin{aligned} \mathcal{Q}_{\mathbf{k},e}(t) \,=\, 1 &- \sin^2(2\theta)\,|U_\mathbf{k}|^2 \,\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) \\ &- \sin^2(2\theta)\,|V_\mathbf{k}|^2 \,\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right) , \quad (16) \end{aligned}$$

with $|U_\mathbf{k}|^2 + |V_\mathbf{k}|^2 = 1$ and

$$|V_\mathbf{k}| = \frac{|\mathbf{k}|}{2\sqrt{\omega_{k,1}\omega_{k,2}}} \left(\sqrt{\frac{\omega_{k,1}+m_1}{\omega_{k,2}+m_2}} - \sqrt{\frac{\omega_{k,2}+m_2}{\omega_{k,1}+m_1}} \right) \quad (17)$$

This result is exact. The differences with respect to the usual formula for neutrino oscillations are in the energy dependence of the amplitudes and in the additional oscillating term. For $|\mathbf{k}| \gg \sqrt{m_1 m_2}$, we have $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0$ and the traditional (Pontecorvo) oscillation formula is recovered.

Finally, it is useful to define the *flavor currents* in the following way:

$$J^\alpha_e(x) \equiv \frac{1}{2}I^\alpha(x) \,+\, J_{f,3}(x) \,, \qquad (18)$$

$$J^\alpha_\mu(x) \equiv \frac{1}{2}I^\alpha(x) \,-\, J_{f,3}(x) \,. \qquad (19)$$

The sum of these two currents gives I^α, i.e. the Noether current density for the total system. These quantities are the natural candidates for a QFT formulation of flavor oscillations in space.

For the bosonic case we can proceed in a similar way. For simplicity, we deal with spin zero stable particles. The Lagrangian for two mixed charged scalar fields is (where necessary, we use a "hat" symbol to distinguish similar quantities from the fermionic case):

$$\begin{aligned} \mathcal{L}(x) &= \partial_\mu \Phi^\dagger_f(x)\partial^\mu \Phi_f(x) \,-\, \Phi^\dagger_f(x)\hat{M}\Phi_f(x) \qquad (20)\\ &= \partial_\mu \Phi^\dagger_m(x)\partial^\mu \Phi_m(x) \,-\, \Phi^\dagger_m(x)\hat{M}_d\Phi_m(x) \qquad (21) \end{aligned}$$

with $\Phi^T_f = (\phi_A, \phi_B)$ being the flavor fields and $\hat{M} = \begin{pmatrix} m^2_A & m^2_{AB} \\ m^2_{AB} & m^2_B \end{pmatrix}$. Those are connected to the free fields $\Phi^T_m = (\phi_1, \phi_2)$ with $\hat{M}_d = diag(m^2_1, m^2_2)$ by a rotation similar to Eq.(2). For an extensive treatment of boson mixing in QFT, see Ref. [16].

Similarly to the fermionic case, we consider the transformation $\Phi'_m(x) = e^{i\alpha_j\tau_j}\,\Phi_m(x)$ with $j = 1, 2, 3$, which gives

$$\delta\mathcal{L}(x) \,=\, i\,\alpha_j\,\Phi^\dagger_m(x)\,[\tau_j, \hat{M}_d]\,\Phi_m(x) \,=\, -\alpha_j\,\partial_\mu\,\hat{J}^\mu_j(x)$$

$$\hat{J}^\mu_{m,j}(x) \,=\, i\,\Phi^\dagger_m(x)\,\tau_j\,\overleftrightarrow{\partial^\mu}\,\Phi_m(x) \quad , \quad j \,=\, 1, 2, 3, \quad (22)$$

where $\overleftrightarrow{\partial^\mu} \equiv \overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu}$.

The charges $\hat{Q}_{m,i}(t) \equiv \int d^3x\, \hat{J}^0_{m,j}(x)$, satisfy the $su(2)$ algebra (at each time t). $2\hat{Q}_{m,2}(t)$ is the generator of the boson mixing transformations whose properties are studied in Refs. [14–16]. The "flavor" vacuum is defined as $|0(t)\rangle_{A,B} \equiv \hat{G}^{-1}_\theta(t)|\hat{0}\rangle_{1,2}$ with $\hat{G}_\theta(t) \equiv e^{i2\theta\hat{Q}_{m,2}(t)}$ and the flavor annihilation operators are $a_{\mathbf{k},A}(t) \equiv \hat{G}^{-1}_\theta(t)\; a_{\mathbf{k},1}\; \hat{G}_\theta(t)$, etc.. As in the fermionic case, these operators involve Bogoliubov coefficients, which now satisfy the hyperbolic relation $|\hat{U}_\mathbf{k}|^2 - |\hat{V}_\mathbf{k}|^2 = 1$ with [14–16]

$$|\hat{V}_\mathbf{k}| \equiv \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}} \right) . \qquad (23)$$

The flavor charges for bosons are:

$$\hat{Q}_\sigma(t) = \int d^3k \left(a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) \,-\, b^\dagger_{-\mathbf{k},\sigma}(t)b_{-\mathbf{k},\sigma}(t) \right) ,$$

where $\sigma = A, B$. We define a flavor "A" state as $|a_A\rangle \equiv a^\dagger_{\mathbf{k},A}(0)|0\rangle_{A,B}$. On this state, the expectation values of the flavor charges give

$$\begin{aligned} \hat{\mathcal{Q}}_\sigma(t) &\equiv \langle a_A|\hat{Q}_\sigma(t)|a_A\rangle \qquad (24)\\ &= \left|\left[a_{\mathbf{k},\sigma}(t), a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 \,-\, \left|\left[b^\dagger_{\mathbf{k},\sigma}(t), a^\dagger_{\mathbf{k},A}(0)\right]\right|^2 . \end{aligned}$$

We also have ${}_{A,B}\langle 0|\hat{Q}_\sigma(t)|0\rangle_{A,B} = 0$ and $\hat{\mathcal{Q}}_{\mathbf{k},A}(t) + \hat{\mathcal{Q}}_{\mathbf{k},B}(t) = 1$. The explicit calculation gives

$$\hat{\mathcal{Q}}_{\mathbf{k},A}(t) = 1 - \sin^2(2\theta)\,|\hat{U}_{\mathbf{k}}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + \sin^2(2\theta)\,|\hat{V}_{\mathbf{k}}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right) . \quad (25)$$

For $|\mathbf{k}|^2 \gg \frac{m_1^2+m_2^2}{2}$ we have $|\hat{V}_{\mathbf{k}}| \to 0$ and the usual result is recovered. Notice the negative sign in front of $|\hat{V}_{\mathbf{k}}|^2$ in contrast with the one for fermions: the oscillating flavor charge for bosons can assume also negative values (of course, charge conservation holds for the overall system of two mixed fields). This happens since in QFT, flavor states are essentially multiparticle ones and one has to abandon the single–particle probabilistic interpretation of QM in favor of a statistical picture.

It is important to stress again that the use of the flavor Hilbert space, i.e. of the Hilbert space constructed on the flavor vacuum, is absolutely crucial in obtaining the above oscillation formulas Eqs.(16), (25). In fact, although the currents and the charges for the flavor fields are defined purely from symmetry considerations on the Lagrangian, the choice of the representation is essential when their expectation values have to be defined. It turns out indeed [12,13,16], that arbitrary mass parameters can be introduced in the expansion of the flavor fields, and that the only quantities free from such arbitrary quantities are the expectation values of the flavor charges on states defined as above in the flavor Hilbert space.

III. CONCLUSIONS

In this paper, we have shown that it is possible to define consistently currents and charges for a system of mixed fields, i.e. for fields with no definite masses. This was done in the case of two flavors, by studying the $SU(2)$ transformations acting on a doublet of fields with different masses: in this way, we naturally recovered the generator of the mixing transformations already introduced in Refs. [9,16] for fermions and bosons, respectively. Our analysis also leads in a straightforward way to the exact QFT oscillation formulas found recently [11,13,16] which exhibit relevant corrections with respect to the usual textbook QM ones.

Our results provide a basis for an exact QFT calculation of oscillation formulas with dependence on space rather than time, a situation which is much closer to the experimental setup. This is one of the aspect which we intend to study in the future.

In the present paper we considered two–flavor mixing for simplicity. The above analysis can be extended to the mixing among an higher number of generations. The case of three–flavor fermion mixing is particularly relevant for the phenomenology (quarks, neutrinos) and such an extension is currently under investigation.

This work has been partially supported by INFN and MURST.

---

[1] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford, 1989.

[2] R. Mohapatra and P. Pal, *Massive Neutrinos in Physics and Astrophysics*, (World Scientific, Singapore, 1991); C. W. Kim and A. Pevsner, *Neutrinos in physics and astrophysics*, (Harwood, 1993).

[3] M. Zralek, Acta Phys. Polon. **B29** (1998) 3925 [hep-ph/9810543].

[4] see for example: Y. Nir, hep-ph/0008226; D. Wyler, hep-ph/0101259, and references therein.

[5] S. M. Bilenky and B. Pontecorvo, *Phys. Rep.* **41** (1978) 225 ; S. M. Bilenkii and S. T. Petcov, Rev. Mod. Phys. **59** (1987) 671; S. M. Bilenky, [hep-ph/0001311].

[6] Super-Kamiokande Collaboration (Y. Fukuda et al.). *Phys. Rev. Lett.* **81** (1998) 1562.

[7] H. Fritzsch and Z. Xing, Prog. Part. Nucl. Phys. **45** (2000) 1; *Phys. Rev.* **D 57** (1998) 594; *Phys. Lett.* **B 413** (1997) 396 ; R. Barbieri, P. Creminelli and A. Romanino, Nucl. Phys. **B559** (1999) 17; R. Barbieri, L. J. Hall, G. L. Kane and G. G. Ross, [hep-ph/9901228].

[8] C. Giunti and C. W. Kim, [hep-ph/0011074].

[9] M. Blasone and G. Vitiello, Annals Phys. **244** (1995) 283.

[10] K. C. Hannabuss and D. C. Latimer, J. Phys. **A33** (2000) 1369.

[11] M. Blasone, P. A. Henning and G. Vitiello, Phys. Lett. **B451** (1999) 140; M. Blasone, in "Erice 1998, From the Planck length to the Hubble radius" 584-593, [hep-ph/9810329].

[12] K. Fujii, C. Habe and T. Yabuki, Phys. Rev. **D59** (1999) 113003; K. Fujii, C. Habe and T. Yabuki, [hep-ph/0102001].

[13] M. Blasone and G. Vitiello, Phys. Rev. **D60** (1999) 111302.

[14] M. Blasone, P.A. Henning and G. Vitiello, in "La Thuile 1996, Results and perspectives in particle physics" 139-152 [hep-ph/9605335].

[15] M. Binger and C. Ji, Phys. Rev. **D60** (1999) 056005.

[16] M. Blasone, A. Capolupo, O. Romei and G. Vitiello, [hep-ph/0102048].

[17] A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer-Verlag, Berlin, 1986).

[18] B. Ancochea, A. Bramon, R. Munoz-Tapia and M. Nowakowski, Phys. Lett. **B389**, 149 (1996).

[19] V. Bargmann, Annals Math. **59** (1954) 1; A. Galindo and P. Pascual, *Quantum Mechanics*, (Springer Verlag, 1990).

[20] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, (McGraw-Hill, New York, 1980).